Quantum Field Theory of Many-body Systems

From the Origin of Sound to an Origin of Light and Electrons

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OXFORD
UNIVERSITY PRESS

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TOPOLOGICAL AND QUANTUM ORDER—BEYOND LANDAU'S THEORIES

In Chapters 3, 4, and 5, we discussed several interacting boson/fermion systems in detail. These simple models illustrate Landau's symmetry-breaking theory (plus RG theory) and Landau's Fermi liquid theory (plus perturbation theory). The two Landau theories explain the behavior of many condensed matter systems and form the foundation of traditional many-body theory. The discussions in the three chapters only scratch the surface of Landau's theories and their rich applications. Readers who want to learn more about Landau's theories (and RG theory and perturbation theory) may find the books by Abrikosov et al. (1975), Ma (1976), Mahan (1990), Negele and Orland (1998), and Chaikin and Lubensky (2000) useful.

Landau's theories are very successful and for a long time people could not find any condensed matter systems that could not be described by Landau's theories. For fifty years, Landau's theories dominated many-body physics and essentially defined the paradigm of many-body physics. After so many years, it has become a common belief that we have figured out all of the important concepts and understood the essential properties of all forms of matter. Many-body theory has reached its end and is a more or less complete theory. The only thing to be done is to apply Landau's theories (plus the renormalization group picture) to all different kinds of systems.

From this perspective, we can understand the importance of the fractional quantum Hall (FQH) effect discovered by Tsui et al. (1982). The FQH effect opened up a new chapter in condensed matter physics. As we have seen in the last chapter, FQH liquids cannot be described by Fermi liquid theory. Different FQH states have the same symmetry and cannot be described by Landau's symmetry-breaking theory. Thus, FQH states are completely beyond the two Landau theories. The existence of FQH liquids indicates that there is a new world beyond the paradigm of Landau's theories. Recent studies suggest that the new paradigm is much richer than the paradigm of Landau's theories. Chapters 7 to 10 of this book are devoted to the new paradigm beyond Landau's theories.

To take a glimpse at the new paradigm of condensed matter physics, we are going to study quantum rotor systems, hard-core boson systems, and quantum spin systems, in addition to the FQH states. These systems are all strongly-correlated

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many-body systems. Usually, the strong correlation in these systems will lead to long-range orders and symmetry breaking. However, we have already discussed long-range order and symmetry breaking, using a boson superfluid and a fermion SDW state as examples. So, to study the world beyond Landau's theories, we concentrate on quantum liquid states (such as the FQH states) that cannot be described by long-range order and symmetry breaking.

These quantum liquid states represent new states of matter that contain a completely new kind of order—topological/quantum order. I will show that the new order may have a deep impact on our understanding of the quantum phase and the quantum phase transition, as well as gapless excitations in the quantum phase. In particular, topological/quantum order might provide an origin for light and electrons (as well as other gauge bosons and fermions) in nature.

In this chapter, we will give a general discussion of topological/quantum order to paint a larger picture. We will use FQH states and Fermi liquid states as examples to discuss some basic issues in topological/quantum order. The problems and issues in topological/quantum order are very different from those in traditional many-body physics. It is very important to understand what the problems are before going into any detailed calculations.

8.1 States of matter and the concept of order

- Matter can have many different states (or different phases). The concept of order is introduced to characterize different internal structures at different states of matter.

- We used to believe that different orders are characterized by their different symmetries.

At sufficiently high temperatures, all matter is in the form of a gas. Gas is one of the simplest states. The motion of an atom in a gas hardly depends on the positions and motion of other molecules. Thus, gases are weakly-correlated systems which contain no internal structure. However, as the temperature is lowered the motion of the atoms becomes more and more correlated. Eventually, the atoms form a very regular pattern and a crystal order is developed. In a crystal, an individual atom can hardly move by itself. Excitations in a crystal always correspond to the collective motion of many atoms (which are called phonons). A crystal is an example of a strongly-correlated state.

With the development of low-temperature technology in around 1900, physicists discovered many new states of matter (such as superconductors and superfluids). These different states have different internal structures, which are called different kinds of orders. The precise definition of order involves phase transition. Two states of a many-body system have the same order if we can smoothly change
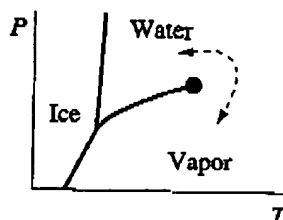
FIG. 8.1. The phase diagram of water.

one state into the other (by smoothly changing the Hamiltonian) without encountering a phase transition (i.e. without encountering a singularity in the free energy). If there is no way to change one state into the other without a phase transition, then the two states will have different orders. We note that our definition of order is a definition of an equivalent class. Two states that can be connected without a phase transition are defined to be equivalent. The equivalent class defined in this way is called the universality class. Two states with different orders can also be said to be two states belonging to different universality classes. According to our definition, water and ice have different orders, while water and vapor have the same order (see Fig. 8.1).

After discovering so many different kinds of order, a general theory is needed to gain a deeper understanding of the states of matter. In particular, we like to understand what makes two orders really different, so that we cannot change one order into the other without encountering a phase transition. The key step in developing the general theory for order and the associated phase and phase transition is the realization that orders are associated with symmetries (or rather, the breaking of symmetries). We find that, when two states have different symmetries, then we cannot change one into the other without encountering a singularity in the free energy (i.e. without encountering a phase transition). Based on the relationship between orders and symmetries, Landau developed a general theory of orders and transitions between different orders (Ginzburg and Landau, 1950; Landau and Lifschitz, 1958). Landau's theory is very successful. Using Landau's theory and the related group theory for symmetries, we can classify all of the 230 different kinds of crystals that can exist in three dimensions. By determining how symmetry changes across a continuous phase transition, we can obtain the critical properties of the phase transition. The symmetry breaking also provides the origin of many gapless excitations, such as phonons, spin waves, etc., which determine the low-energy properties of many systems (Nambu, 1960; Goldstone, 1961). Many of the properties of those excitations, including their gaplessness, are directly determined by the symmetry. Introducing order parameters associated with symmetries,

Ginzburg and Landau developed Ginzburg–Landau theory, which became the stan-dard theory for phase and phase transition. As Landau's symmetry-breaking theory has such a broad and fundamental impact on our understanding of matter, it became a corner-stone of condensed matter theory. The picture painted by Lan-dau's theory is so satisfactory that one starts to have a feeling that we understand, at least in principle, all kinds of orders that matter can have.

8.2 Topological order in fractional quantum Hall states

• The FQH state opened up a new chapter in condensed matter physics, because it was the second state to be discovered by experimentalists that could not be characterized by symmetry breaking and local order parameters.
• The FQH states contain a new kind of order — topological order.

However, nature never ceases to surprise us. With advances in semiconduc-tor technology, physicists learnt how to confine electrons on an interface between two different semiconductors, and hence made a two-dimensional electron gas (2DEG). In 1982, Tsui *et al.* (1982) put a 2DEG under strong magnetic fields and discovered a new state of matter—the FQH liquid (Laughlin, 1983). As the temperatures are low and the interaction between the electrons is strong, the FQH state is a strongly-correlated state. However, such a strongly-correlated state is not a crystal, as people had originally expected. It turns out that the strong quantum fluctuations of electrons, due to their very small mass, prevent the formation of a crystal. Thus, the FQH state is a quantum liquid. (A crystal can melt in two ways, namely by thermal fluctuations as we raise temperatures, which leads to an ordinary liquid, or by quantum fluctuations as we reduce the mass of the particles, which leads to a quantum liquid.)

As we have seen in the last chapter, quantum Hall liquids have many amazing properties. A quantum Hall liquid is more 'rigid' than a solid (a crystal), in the sense that a quantum Hall liquid cannot be compressed. Thus, a quantum Hall liquid has a fixed and well-defined density. When we measure the electron density in terms of the filling fraction, defined by

$$\nu = \frac{\text{density of electron}}{\text{density of magnetic flux quanta}},$$

we find that all of the discovered quantum Hall states have densities such that the filling fractions are given exactly by some rational numbers, such as $\nu = 1, 1/3, 2/3, 2/5, \ldots$. Knowing that FQH liquids exist only at certain magical filling fractions, one cannot help but guess that FQH liquids should have some internal orders or 'patterns'. Different magical filling fractions should be due to these dif-ferent internal 'patterns'. However, the hypothesis of internal 'patterns' appears
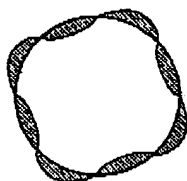
FIG. 8.2. A particle wave on a circle has a quantized wavelength.

to have one difficulty—the FQH states are liquids, and how can liquids have any internal 'patterns'?

To gain some intuitive understanding of the internal order in FQH states, let us try to visualize the quantum motion of electrons in an FQH state. We know that a particle also behaves like a wave, according to quantum physics. Let us first consider a particle moving on a circle with momentum p. Such a particle corresponds to a wave with wavelength $\lambda = h/p$, where h is the Planck constant. Only waves that can fit into the circle are allowed (i.e. the circle must contain an integer number of wavelengths) (see Fig. 8.2). Thus, due to quantum physics, the motion of a particle on a circle is highly restricted (or quantized), and only certain discrete values of momentum are allowed. Such a quantization condition can be viewed in a more pictorial way. We may say that the particle dances around the circle in steps, with a step length given by the wavelength. The quantization condition requires that the particle always takes an integer number of steps to go around the circle.

Now let us consider a single electron in a magnetic field. Under the influence of the magnetic field, the electron always moves along circles (which are called cyclotron motions). In quantum physics, only certain discrete cyclotron motions are allowed due to the wave property of the particle. The quantization condition is such that the circular orbit of an allowed cyclotron motion contains an integer number of wavelengths. We may say that an electron always takes an integer number of steps to go around the circle. If the electron takes n steps around the circle, then we say that the electron is in the nth Landau level. The electrons in the first Landau level have the lowest energy, and the electron will stay in the first Landau level at low temperatures.

When we have many electrons to form a 2DEG, electrons not only do their own cyclotron motion in the first Landau level, but they also go around each other and exchange places. These additional motions are also subject to the quantization condition. For example, an electron must take integer steps to go around another electron. As electrons are fermions, exchanging two electrons introduces a minus sign into the wave function. Also, exchanging two electrons is equivalent to moving one electron half-way around the other electron. Thus, an electron must take

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half-integer steps to go half-way around another electron. (The half-integer steps
introduce a minus sign into the electron wave function.) In other words, an elec-
tron must take an odd number of steps to go around another electron. Electrons in
an FQH state not only move in a way that satisfies the quantization condition, but
they also try to stay away from each other as much as possible, due to the strong
Coulomb repulsion and the Fermi statistics between electrons. This means that an
electron tries to take more steps to go around another electron, if possible.

Now we see that, despite the absence of crystal order, the quantum motions
of electrons in an FQH state are highly organized. All of the electrons in an FQH
state dance collectively, following strict dancing rules.

1. All of the electrons do their own cyclotron motion in the first Landau level,
   i.e. they take one step to go around the circle.

2. An electron always takes an odd number of steps to go around another
   electron.

3. Electrons try to stay away from each other, i.e. they try to take as many steps
   as possible to go around another electron.

If every electron follows these strict dancing rules, then only one unique global
dancing pattern is allowed. Such a dancing pattern describes the internal quantum
motion in the FQH state. It is this global dancing pattern that corresponds to the
internal order in the FQH state. Different FQH states are distinguished by their
different dancing patterns.

A more precise mathematical description of the quantum motion of electrons
outlined above is given by the famous Laughlin wave function (Laughlin, 1983)

$$\Psi_m = \left[\prod (z_i - z_j)^m \right] e^{-\frac{1}{4l_B^2} \sum |z_i|^2}$$

where m is an odd integer and $z_j = x_j + iy_j$ is the coordinate of the jth electron.
Such a wave function describes a filling fraction $\nu = 1/m$ FQH state. We see that
the wave function vanishes as $z_i \to z_j$, so that the electrons do not like to stay
close to each other. Also, the wave function changes its phase by $2\pi m$ as we move
one electron around another. Thus, an electron always takes m steps to go around
another electron in the Laughlin state.

We would like to stress that the internal orders (i.e. the dancing patterns) of
FQH liquids are very different from the internal orders in other correlated systems,
such as crystals, superfluids, etc. The internal orders in the latter systems can be
described by order parameters associated with broken symmetries. As a result,
the ordered states can be described by the Ginzburg–Landau effective theory. The
internal order in FQH liquids is a new kind of ordering which cannot be described

by long-range orders associated with broken symmetries.[51] In 1989, the concept of 'topological order' was introduced to describe this new kind of ordering in FQH liquids (Wen, 1990, 1995).

We would like to point out that topological orders are general properties of any states at zero temperature with a finite energy gap. Non-trivial topological orders not only appear in FQH liquids, but they also appear in spin liquids at zero temperature. In fact, the concept of topological order was first introduced (Wen, 1990) in a study of chiral spin liquids (Kalmeyer and Laughlin, 1987; Khveshchenko and Wiegmann, 1989; Wen et al., 1989). In addition to chiral spin liquids, non-trivial topological orders were also found in anyon superfluids (Chen et al., 1989; Fetter et al., 1989; Wen and Zee, 1991) and short-ranged resonating valence bound states for spin systems (Kivelson et al., 1987; Rokhsar and Kivelson, 1988; Read and Chakraborty, 1989; Read and Sachdev, 1991; Wen, 1991a). The FQH liquid is not even the first experimentally observed state with non-trivial topological orders. That honor goes to the superconducting state discovered in 1911 (Onnes, 1911; Bardeen et al., 1957). In contrast to a common point of view, a superconducting state, with dynamical electromagnetic interactions, cannot be characterized by broken symmetries. It has neither long-range orders nor local order parameters. A superconducting state contains non-trivial topological orders. It is fundamentally different from a superfluid state (Coleman and Weinberg, 1973; Halperin et al., 1974; Fradkin and Shenker, 1979).

It is instructive to compare FQH liquids with crystals. FQH liquids are similar to crystals in the sense that they both contain rich internal patterns (or internal orders). The main difference is that the patterns in the crystals are static, related to the positions of atoms, while the patterns in QH liquids are 'dynamic', associated with the ways that electrons 'dance' around each other. However, many of the same questions for crystal orders can also be asked and should be addressed for topological orders. We know that crystal orders can be characterized and classified by symmetries. Thus, one important question is how do we characterize and classify the topological orders? We also know that crystal orders can be measured by X-ray diffraction. The second important question is how do we experimentally measure the topological orders?

In the following, we are going to discuss topological orders in FQH states in more detail. It turns out that FQH states are quite typical topologically-ordered states. Many other topologically-ordered states share many similar properties with FQH states.

8.2.1 Characterization of topological orders

Any new concept in physics must be introduced (or defined) via quantities that can be measured by experiments. To define a physical quantity or concept is to design an experiment.

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[51] Although it was suggested that the internal structures of Laughlin states can be characterized by an 'off-diagonal long-range order' (Girvin and MacDonald, 1987), the operator that has long-range order itself is not a local operator. For local operators, there is no long-range order and there are no symmetry-breaking Laughlin states.

• The concept of topological order is characterized by ground-state degeneracy which is robust against perturbation even if it can break all of the symmetries.

In the above, the concept of topological order (the dancing pattern) is introduced through the ground-state wave function. This is not quite correct because the ground-state wave function is not universal. To establish a new concept, such as topological order, one needs to find physical characterizations or measurements of topological orders. In other words, one needs to find universal quantum numbers that are robust against any perturbation, such as changes in interactions, effective mass, etc., but which can take different values for different classes of FQH liquids. The existence of such quantum numbers implies the existence of topological orders.

One way to show the existence of topological orders in FQH liquids is to study their ground-state degeneracies (in the thermodynamical limit). FQH liquids have a very special property. Their ground-state degeneracy depends on the topology of space (Haldane, 1983; Haldane and Rezayi, 1985). For example, the $\nu = \frac{1}{q}$ Laughlin state has q^g degenerate ground states on a Riemann surface of genus g. The ground-state degeneracy in FQH liquids is *not* a consequence of the symmetry of the Hamiltonian. The ground-state degeneracy is robust against arbitrary perturbations (even impurities that break all of the symmetries in the Hamiltonian) (Wen and Niu, 1990). The robustness of the ground-state degeneracy indicates that the internal structures that give rise to ground-state degeneracy are universal and robust, hence demonstrating the existence of universal internal structures—topological orders.

To understand the topological degeneracy of FQH ground states, we consider a $\nu = 1/m$ Laughlin state on a torus. We will use two methods to calculate the ground-state degeneracy. In the first method, we consider the following tunneling process. We first create a quasiparticle–quasihole pair. Then we bring the quasiparticle all the way around the torus. Finally, we annihilate the quasiparticle–quasihole pair and go back to the ground state. Such a tunneling process produces an operator that maps ground states to ground states. Such an operator is denoted by U_x if the quasiparticle goes around the torus in the x direction, and U_y if it goes around the torus in the y direction (see Fig. 8.3(a,b)). Then, the four tunnelings in the x, y, $-x$, and $-y$ directions generate $U_y^{-1} U_x^{-1} U_y U_x$ (see Fig. 8.3(c)). We note that the path of the above four tunnelings can be deformed into two linked loops (see Fig. 8.3(d)). The two linked loops correspond to moving one quasiparticle around the other. It gives rise to a phase $e^{2i\theta}$, where θ is the statistical angle of the quasiparticle. For the $1/m$ Laughlin state, we have $\theta = \pi/m$. Therefore, we have
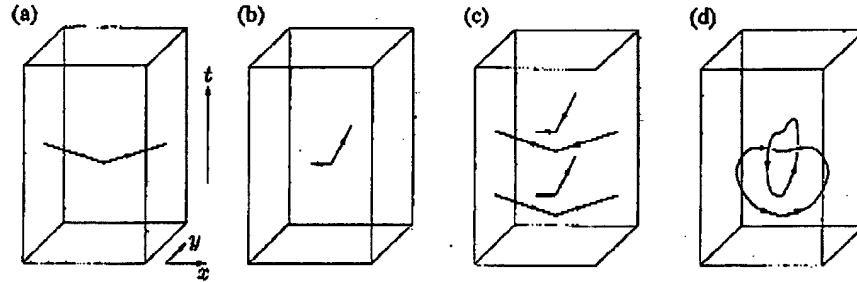
$$U_y^{-1} U_x^{-1} U_y U_x = e^{i2\pi/m}$$

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FIG. 8.3. (a) Tunneling in the x direction generates U_x. (b) Tunneling in the y direction generates U_y. (c) The four tunnelings in the x, y, $-x$, and $-y$ directions generate $U_y^{-1}U_x^{-1}U_yU_x$. (d) The above four tunnelings can be deformed into two linked loops.

As $U_{x,y}$ acts within the ground states, the ground states form the representation of the above algebra. The algebra has only one m-dimensional irreducible representation. Thus, the $1/m$ Laughlin state has m × integer number of degenerate ground states. This approach allows us to see the direct connection between the quasiparticle statistics and the ground-state degeneracy.

The second way to calculate the ground-state degeneracy is to use the effective theory (7.3.11). The degenerate ground states arise from the following collective fluctuations:

$$a_i(t, x, y) = \theta_i(t)/L, \quad i = x, y \qquad (8.2.1)$$

where L is the size of the torus in the x and y directions. All of the other fluctuations generate a nonzero 'magnetic' field $b = f_{xy}$ and have a finite energy gap, as one can see from the classical equation of motion. The following Lagrangian describing the dynamics of the collective excitations in eqn (8.2.1) can be obtained by substituting eqn (8.2.1) into eqn (7.3.11):

$$L = -\frac{m}{4\pi}(\dot{\theta}_x\theta_y - \dot{\theta}_y\theta_x) + \frac{1}{2g_1}\dot{\theta}_i^2. \qquad (8.2.2)$$

Since the charge of a_μ is quantized as an integer, the gauge transformation $U(x, y)$ that acts on the quasiparticle field, $\psi_q \to U\psi_q$, must be a periodic function on the torus. Thus, the gauge transformation must have the form $U(x, y) = \exp\left(2\pi i(\frac{nx}{L} + \frac{my}{L})\right)$, where n and m are integers. As the a_μ charge of ψ_q is 1, such a gauge transformation changes the gauge field a_i to $a_i' = a_i - iU^{-1}\partial_i U$ as follows:

$$(a_x', a_y') = (a_x + \frac{2\pi n}{L}, a_y + \frac{2\pi m}{L}) \qquad (8.2.3)$$

Equation (8.2.3) implies that (θ_x, θ_y) and $(\theta_x + 2\pi n, \theta_y + 2\pi m)$ are gauge equivalent and should be identified. The gauge-inequivalent configurations are given by

a point on a torus $0 < \theta_i < 2\pi$. As a result, the Lagrangian (8.2.2) describes a particle with unit charge moving on a torus parametrized by (θ_x, θ_y).

The first term in eqn (8.2.2) indicates that there is a uniform 'magnetic' field $B = m/2\pi$ on the torus. The total flux passing through the torus is equal to $2\pi \times m$. The Hamiltonian of eqn (8.2.2) is given by

$$H = \frac{g_1}{2}\left[-(\partial_{\theta_x} - iA_{\theta_x})^2 - (\partial_{\theta_y} - iA_{\theta_y})^2\right]. \qquad (8.2.4)$$

The energy eigenstates of eqn (8.2.4) form Landau levels. The gap between the Landau levels is of order g_1, which is independent of the size of the system. The number of states in the first Landau level is equal to the number of flux quanta passing through the torus, which is m in our case. Thus, the ground-state degeneracy of the $1/m$ Laughlin state is m.

To understand the robustness of the ground-state degeneracy, let us add an arbitrary perturbation to the electron Hamiltonian. Such a perturbation will cause a change in the effective Lagrangian, $\delta\mathcal{L}(a_\mu)$. The key here is that $\delta\mathcal{L}(a_\mu)$ only depends on a_μ through its field strength. In other words, $\delta\mathcal{L}$ is a function of e and b. As the collective fluctuation in eqn (8.2.1) is a pure gauge locally, e and b, and hence $\delta\mathcal{L}$, do not depend on θ_i. The $\delta\mathcal{L}(a_\mu)$ cannot generate any potential terms $V(\theta_i)$ in the effective theory of θ_i. It can only generate terms that only depend on $\dot{\theta}_i$. Such a correction renormalizes the value of g_1. It cannot lift the degeneracy.

More general FQH states are described by eqn (7.3.18). When K is diagonal, the above result implies that the ground-state degeneracy is $\det(K)$. It turns out that, for a generic K, the ground-state degeneracy is also given by $\det(K)$. On a Riemann surface of genus g, the ground-state degeneracy becomes $(\det(K))^g$.

We see that in a compact space the low-energy physics of FQH liquids are very unique. There are only a finite number of low-energy excitations (i.e. the degenerate ground states), yet the low-energy dynamics are non-trivial because the ground-state degeneracy depends on the topology of the space. Such special low-energy dynamics, which depend only on the topology of the space, are described by the so-called topological field theory, which was studied intensively in the high-energy physics community (Witten, 1989; Elitzur et al., 1989; Fröhlich and King, 1989). Topological theories are effective theories for FQH liquids, just as the Ginzburg–Landau theory is for superfluids (or other symmetry-broken phases).

The dependence of the ground-state degeneracy on the topology of the space indicates the existence of some kind of long-range order (the global dancing pattern mentioned above) in FQH liquids, despite the absence of long-range correlations for all local physical operators. In some sense, we may say that FQH liquids contain hidden long-range orders.

8.2.2   Classification of topological orders

- It is important to understand the mathematical framework behind topological order. (Just like it is important to understand group theory—the mathematical framework behind the symmetry-breaking orders.)
- The understanding of the mathematical framework will allow us to classify all of the possible topological orders.

A long-standing problem has been how to label and classify the rich topological orders in FQH liquids. We are able to classify all crystal orders because we know that the crystal orders are described by a symmetry group. However, our understanding of topological orders in FQH liquids is very poor, and the mathematical structure behind topological orders is unclear.

Nevertheless, we have been able to find a simple and unified treatment for a class of FQH liquids—abelian FQH liquids (Blok and Wen, 1990a,b; Read, 1990; Fröhlich and Kerler, 1991; Fröhlich and Studer, 1993). Laughlin states represent the simplest abelian FQH states that contain only one component of incompressible fluid. More general abelian FQH states with a filling fraction such as $\nu = 2/5, 3/7, ...$ contain several components of incompressible fluids, and have more complicated topological orders. The topological orders (or the dancing patterns) in the abelian FQH state can also be described by the dancing steps. The dancing patterns can be characterized by an integer symmetric matrix K and an integer charge vector q. An entry of q, q_i, is the charge (in units of e) carried by the particles in the ith component of the incompressible fluid. An entry of K, K_{ij}, is the number of steps taken by a particle in the ith component to go around a particle in the jth component. In the (K, q) characterization of FQH states, the $\nu = 1/m$ Laughlin state is described by $K = m$ and $q = 1$, while the $\nu = 2/5$ abelian state is described by $K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ and $q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

All of the physical properties associated with the topological orders can be determined in terms of K and q. For example, the filling fraction is simply given by $\nu = q^{T} K^{-1} q$ and the ground-state degeneracy on the genus g Riemann surface is $\det(K)^g$. All of the quasiparticle excitations in this class of FQH liquids have abelian statistics, which leads to the name abelian FQH liquids.

The above classification of FQH liquids is not complete. Not every FQH state is described by K-matrices. In 1991, a new class of FQH states—non-abelian FQH states—was proposed (Moore and Read, 1991; Wen, 1991b). A non-abelian FQH state contains quasiparticles with non-abelian statistics. The observed filling fraction $\nu = 5/2$ FQH state (Willett et al., 1987) is very likely to be one such state (Haldane and Rezayi, 1988a,b; Greiter et al., 1991; Read and Green, 2000). Many studies (Moore and Read, 1991; Blok and Wen, 1992; Iso et al., 1992; Cappelli et al., 1993; Wen et al., 1994) have revealed a connection between the topological

orders in FQH states and conformal field theories. However, we are still quite far from a complete classification of all possible topological orders in non-abelian states.

8.2.3 Edge excitations—a practical way to measure topological orders

The edge excitations for FQH states play a similar role to X-rays for crystals. We can use edge excitations to experimentally probe the topological orders in FQH states. In other words, compared to ground-state degeneracy, edge excitations provide a more complete definition of topological orders.

Topological degeneracy of the ground states only provides a partial characterization of topological orders. Different topological orders can sometimes lead to the same ground-state degeneracy. The issue here is whether we have a more complete characterization/measurement of topological orders. Realizing that the topological orders cannot be characterized by local order parameters and long-range correlations of local operators, it seems difficult to find any methods to characterize topological order. Amazingly, FQH states find a way out in an unexpected fashion. The bulk topological orders in FQH states can be characterized/measured by edge excitations (Wen, 1992). This phenomenon of two-dimensional topological orders being encoded in one-dimensional edge states shares some similarities with the holomorphic principle in superstring theory and quantum gravity ('t Hooft, 1993; Susskind, 1995).

FQH liquids as incompressible liquids have a finite energy gap for all of their bulk excitations. However, FQH liquids of finite size always contain one-dimensional gapless edge excitations, which is another unique property of FQH fluids. The structures of edge excitations are extremely rich, which reflects the rich bulk topological orders. Different bulk topological orders lead to different structures of edge excitations. Thus, we can study and measure the bulk topological orders by studying the structures of edge excitations.

As we have seen in the last chapter that, due to the non-trivial bulk topological order, the electrons at the edges of (abelian) FQH liquids form a new kind of correlated state—chiral Luttinger liquids (Wen, 1992). The electron propagator in chiral Luttinger liquids develops an anomalous exponent: $\langle c^\dagger(t,x)c(0)\rangle \propto (x-vt)^{-g}$, $g \neq 1$. (For Fermi liquids, we have $g = 1$.) The exponent g, in many cases, is a topological quantum number which does not depend on detailed properties of the edges. Thus, g is a new quantum number that can be used to characterize the topological orders in FQH liquids. Many experimental groups have successfully measured the exponent g through the temperature dependence of tunneling conductance between two edges (Milliken et al., 1995; Chang et al., 1996), which was predicted to have the form $\sigma \propto T^{2g-2}$ (Wen, 1992). This experiment demonstrates
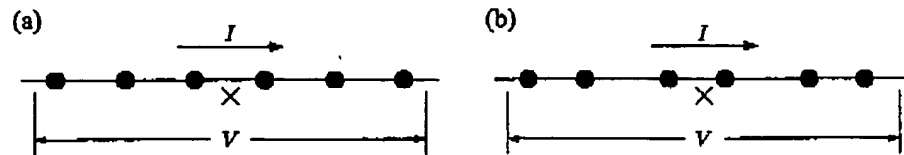
FIG. 8.4. A one-dimensional crystal passing an impurity will generate narrow-band noise in the voltage drop.
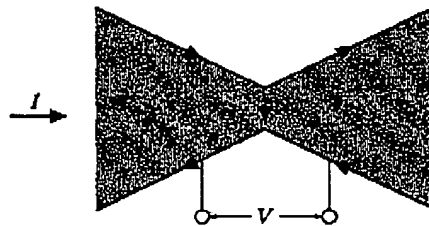

FIG. 8.5. An FQH fluid passing through a constriction will generate narrow-band noises due to the back-scattering of the quasiparticles.

the existence of new chiral Luttinger liquids and opens the door to the experimental study of the rich internal and edge structures of FQH liquids.

The edge states of non-abelian FQH liquids form more exotic one-dimensional correlated systems which have not yet been named. These edge states were found to be closely related to conformal field theories in $1 + 1$ dimensions (Wen et al., 1994).

We know that crystal orders can be measured by X-ray diffraction experiments. In the following, we would like to suggest that the topological orders in FQH liquids can be measured (in principle) through a noise spectrum in an edge transport experiment. Let us first consider a one-dimensional crystal driven through an impurity (see Fig. 8.4(a)). Due to the crystal order, the voltage across the impurity has a narrow-band noise at a frequency of $f = I/e$ if each unit cell has only one charged particle. More precisely, the noise spectrum has a singularity, i.e. $S(f) \sim A\delta(f - \frac{I}{e})$. If each unit cell contains two charged particles (see Fig. 8.4(b)), then we will see an additional narrow-band noise at $f = I/2e$, so that $S(f) \sim B\delta(f - \frac{I}{2e}) + A\delta(f - \frac{I}{e})$. In this example, we see that the noise spectrum allows us to measure crystal orders in one-dimension. A similar experiment can also be used to measure topological orders in FQH liquids. Let us consider an FQH sample with a narrow constriction (see Fig. 8.5). The constriction induces a back-scattering through quasiparticle tunneling between the two

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edges. The back-scattering causes a noise in the voltage across the constriction. In
the weak-back-scattering limit, the noise spectrum contains singularities at certain
frequencies, which allows us to measure the topological orders in the FQH liq-
uids.[52] To be more specific, the singularities in the noise spectrum have the form
(see eqn (7.4.49))

$$S(f) \sim \sum_a C_a |f - f_a|^{\gamma_a} \qquad (8.2.5)$$

The frequencies and the exponents of the singularities (f_a, γ_a) are determined by
the topological orders. For the abelian state characterized by the matrix K and the
charge vector q, the allowed values of the pair (f_a, γ_a) are given by

$$f_a = \frac{I}{e\nu} q^\top K^{-1} l, \qquad \gamma_a = 2 l^\top K^{-1} l - 1 \qquad (8.2.6)$$

where $l^\top = (l_1, l_2, ...)$ is an arbitrary integer vector and $\nu = q^\top K^{-1} q$ is the filling
fraction. The singularities in the noise spectrum are caused by quasiparticle tun-
neling between the two edges. The frequency of the singularity f_a is determined
by the electric charge of the tunneling quasiparticle Q_q, i.e. $f_a = \frac{I}{e} \frac{Q_q}{e\nu}$. The expo-
nent γ_a is determined by the statistics of the tunneling quasiparticle θ_q, namely
$\gamma = 2\frac{|\theta_q|}{\pi} - 1$. Thus, the noise spectrum measures the charge and the statistics of
the allowed quasiparticles, which in turn determines the topological orders in FQH
states.

8.3 Quantum orders

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[52] The discussion presented here applies only to the FQH states whose edge excitations all propa-
gate in the same direction. This requires, for abelian states, all of the eigenvalues of K to have the
same sign.

The topological order, by definition, only describes the internal order of gapped quantum states. Here we make a leap of faith. We will assume that the gap is not important and that gapless quantum states can also contain orders that cannot be described by symmetry and long-range correlations. We will call the non-symmetry-breaking order in quantum ground states the quantum order.

If you believe in this line of thinking, then the only things that need to be done are to show that quantum orders do exist, and to find mathematical descriptions (or symbols) that characterize the quantum orders. We will show that quantum orders do exist in Section 8.3.2 and in Chapter 10. As quantum orders cannot be characterized by broken symmetries and order parameters, we need to develop a new theory to describe quantum orders. At present, we do not have a complete theory that can describe all possible quantum orders. However, in Chapter 9 we manage to find a mathematical object—the projective symmetry group (PSG)—that can describe a large class of quantum orders.

One may ask, why do we need to introduce the new concept of quantum order? What use can it have? To answer such a question, we would like to ask, why do we need the concept of symmetry breaking? Is the symmetry-breaking description useful? Symmetry breaking is useful because it leads to a classification of crystal orders (such as the 230 different crystals in three dimensions), and it determines the structure of low-energy excitations without the need to know the details of a system (such as three branches of phonons from three broken translational symmetries in a solid) (Nambu, 1960; Goldstone, 1961). The quantum order and its PSG description are useful in the same sense; a PSG can classify different quantum states that have the same symmetry (Wen, 2002c), and quantum orders determine the structure of low-energy excitations without the need to know the details of a system (Wen, 2002a,c; Wen and Zee, 2002). The main difference between symmetry-breaking orders and quantum orders is that symmetry-breaking orders generate and protect gapless Nambu–Goldstone modes (Nambu, 1960; Goldstone, 1961), which are scalar bosonic excitations, while quantum orders can generate and protect gapless gauge bosons and gapless fermions. Fermion excitations can even emerge in pure local bosonic models, as long as the boson ground state has a proper quantum order.

One way to visualize quantum order is to view quantum order as a description of the pattern of the quantum entanglement in a many-body ground state. Different patterns of entanglement give rise to different quantum orders. The fluctuations of entanglement correspond to collective excitations above a quantum-ordered state. We will see that these collective excitations can be gauge bosons and fermions.

The concept of topological/quantum order is also useful in the field of quantum computation. People have been designing different kinds of quantum-entangled states to perform different computing tasks. When the number of qubits becomes larger and larger, it is more and more difficult to understand the pattern of quantum

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entanglements. One needs a theory to characterize different quantum entanglements in many-qubit systems. The theory of topological/quantum order (Wen, 1995, 2002c) is just such a theory. Also, the robust topological degeneracy in topologically-ordered states discovered by Wen and Niu (1990) can be used in fault-tolerant quantum computation (Kitaev, 2003).

In the following, we will discuss the connection between quantum phase transitions and quantum orders. Then we will use the quantum phase transitions in free fermion systems to study the quantum orders there.

8.3.1   Quantum phase transitions and quantum orders

- Quantum phase transitions are defined by the singularities of the ground-state energy as a function of the parameters in the Hamiltonian.

Classical orders can be studied through classical phase transitions. Classical phase transitions are marked by singularities in the free-energy density f. The free-energy density can be calculated through the partition function as follows:

$$f = -\frac{T \ln Z}{V_{space}}, \quad Z = \int \mathcal{D}\phi \, e^{-\beta \int dx \, h(\phi)} \tag{8.3.1}$$

where $h(\phi)$ is the energy density of the classical system and V_{space} is the volume of space.

Similarly, to study quantum orders we need to study quantum phase transitions at zero temperature $T = 0$. Here the energy density of the ground state plays the role of the free-energy density. A singularity in the ground-state-energy density marks a quantum transition. The similarity between the ground-state-energy density and the free-energy density can be clearly seen in the following expression for the energy density of the ground state:

$$\rho_E = i \frac{\ln Z}{V_{space-time}}, \quad Z = \int \mathcal{D}\phi \, e^{i \int dx \, dt \, \mathcal{L}(\phi)} \tag{8.3.2}$$

where $\mathcal{L}(\phi)$ is the Lagrangian density of the quantum system and $V_{space-time}$ is the volume of space–time. Comparing eqns (8.3.1) and (8.3.2), we see that a classical system is described by a path integral of a positive functional, while a quantum system is described by a path integral of a complex functional. In general, a quantum phase transition, marked by a singularity of the path integral of a complex functional, can be more general than classical phase transitions that are marked by a singularity of the path integral of a positive functional.
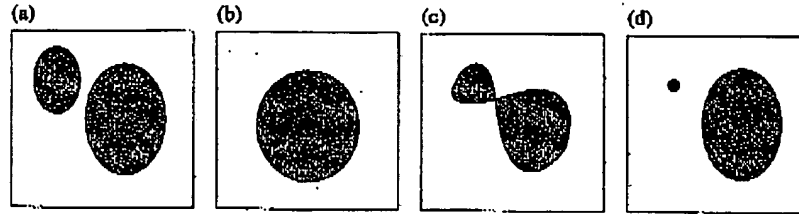
FIG. 8.6. The two sets of oriented Fermi surfaces in (a) and (b) represent two different quantum orders. The two possible transition points between the two quantum orders in (a) and (b) are described by the Fermi surfaces in (c) and (d).

8.3.2 Quantum orders and quantum transitions in free fermion systems

- Free fermion systems contain quantum phase transitions that do not change any symmetry, indicating that free fermion systems contain non-trivial quantum orders.
- Different quantum orders in free fermion systems are classified by the topologies of a Fermi surface.

Let us consider a free fermion system with only the translational symmetry and the $U(1)$ symmetry from the fermion number conservation. The Hamiltonian has the form

$$H = \sum_{\langle ij \rangle} \left(c_i^\dagger t_{ij} c_j + h.c. \right)$$

with $t_{ij}^* = t_{ji}$. The ground state is obtained by filling every negative energy state with one fermion. In general, the system contains several pieces of Fermi surfaces.

To understand the quantum order in the free fermion ground state, we note that the topology of the Fermi surfaces can change in two ways as we continuously change t_{ij}: a Fermi surface can shrink to zero (Fig. 8.6(d)); and two Fermi surfaces can join (Fig. 8.6(c)). When a Fermi surface is about to disappear in a d-dimensional system, the ground-state-energy density has the form

$$\rho_E = \int \frac{d^d k}{(2\pi)^d} (k \cdot M \cdot k - \mu)\Theta(-k \cdot M \cdot k + \mu) + \ldots$$

where '...' represents the non-singular contribution and the symmetric matrix M is positive (or negative) definite. We find that the ground-state-energy density has a singularity at $\mu = 0$, i.e. $\rho_E = c\mu^{(2+d)/2}\Theta(\mu) + \ldots$, where $\Theta(x > 0) = 1$ and $\Theta(x < 0) = 0$. When two Fermi surfaces are about to join, the singularity is still determined by the above equation, but now M has both negative and positive eigenvalues. The ground-state-energy density has a singularity of the form $\rho_E = c\mu^{(2+d)/2}\Theta(\mu) + \ldots$ when d is odd and $\rho_E = c\mu^{(2+d)/2}\log|\mu| + \ldots$ when d is even.
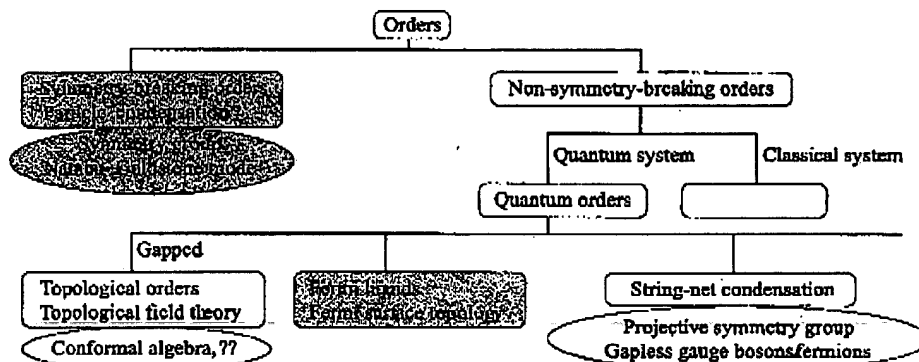
FIG. 8.7. A new classification of orders. The phases in the shaded boxes can be described Landau's theories. Other phases are beyond Landau's theories.

The singularity of the ground-state-energy density at $\mu = 0$ indicates a quantum phase transition. This kind of transition was first studied by Lifshitz (1960). Clearly, there is no change of symmetry across the transition and there is no local order parameters to characterize the phases on the two sides of the transition. This suggests that the two states can only be distinguished by their quantum orders. As the $\mu = 0$ point is exactly the place where the topology of the Fermi surface changes, we find that the topology of the Fermi surface is a 'quantum number' that characterizes the quantum order in a free fermion system (see Fig. 8.6). A change in the topology signals a *continuous* quantum phase transition that changes the quantum order.

Problem 8.3.1.

Consider a two-dimensional spin-1/2 free electron system. As we change the chemical potential μ, the system undergoes a quantum phase transition, as illustrated in Fig. 8.6(d). Find the singular behavior of the spin susceptibility near the transition point μ_c.

8.4 A new classification of orders

* Quantum orders have many classes. FQH states and free fermion systems represent only two of the many classes of quantum orders.

The concept of topological/quantum order allows us to have a new classification of orders, as illustrated in Fig. 8.7. According to this classification, a quantum order is simply a non-symmetry-breaking order in a quantum system, and a topological order is simply a quantum order with a finite energy gap.

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From the FQH states and the Fermi liquid states discussed above, we see that quantum order can be divided into several different classes. The FQH states and the free fermion systems only provide two examples of quantum orders. In the next few chapters, we will study quantum orders in some strongly-correlated systems. We will show that these quantum orders belong to a different class, which is closely related to a condensation of nets of strings in the correlated ground state. Chapter 9 studies and classifies this class of quantum order using a projective construction (i.e. the slave-boson approach). Chapter 10 relates the quantum-ordered state studied in Chapter 9 to string-net-condensed states.